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Exact Solution of Petrov Type {3, 1} Metric via Time Dependent Quasi-Maxwell Equations

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Abstract In this paper, we show that how to apply the time dependent quasi-Maxwell equations for exact solution of Einstein field equations in vacuum for Petrov type {3, 1} metric.

Keywords Petrov type {3, 1} metric · Time-dependent quasi-Maxwell equations · Exact solution

1 Introduction

In threading point of view, splitting of spacetime is introduced by a family of timelike congruences with unit tangent vector field, may be interpreted as the world-lines of a family of observers, and it defines a local time direction plus a local space through its orthogonal subspace in the tangent space. Let (M, g) be a 4-dim manifold of a stationary spacetime. Next, we can construct a 3-dim orbit manifold as $\bar{M} = \frac{M}{G}$ with projected metric tensor[1] γ_{ij} by the smooth map $\eta : M \to \bar{M}$ in which $\eta(p)$ denotes the orbit of the timelike Killing vector $\frac{\partial}{\partial t}$ at the point $p \in M$ and G is 1-dim group of transformations generated by the timelike Killing vector of the spacetime under consideration, [1, 2]. The threading decomposition leads to the following splitting of the spacetime distance element, [3, 4]:

$$ds^2 = \mathrm{g}_{\alpha\beta} dx^\alpha dx^\beta = h(dt - g_i dx^i)^2 - \gamma_{ij} dx^i dx^j, \tag{1}$$

where $\gamma_{ij} = -\mathrm{g}_{ij} + h g_i g_j$ with $g_i = -\frac{\mathrm{g}_{0i}}{h}$ and $h = \mathrm{g}_{00}$. In a spacetime with time dependent metric tensor (1), the gravitational Lorentz force acting on a test particle whose mass m_0 due to time dependent gravitoelectromagnetism[2] fields as measured by threading observers

[1] Note that the Greek indices run from 0 to 3 while the Latin indices take the values 1 to 3.

[2] The gravitoelectromagnetism refers to a set of analogies between Maxwell equations and a reformulation of the Einstein field equations in general relativity, [5].

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is described by the following equation, we use geometrical units with $c = G = 1$, [3]:

$$\mathbf{F} = -\frac{{}^*\partial \mathbf{p}}{\partial t} + \frac{m_0}{\sqrt{1-v^2}}\{{}^*\mathbf{E}_g + \mathbf{v} \times {}^*\mathbf{B}_g\}, \tag{2}$$

where $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{h}}\partial_0$ while ∂_0 indicate the time derivative, $\mathbf{v}$ is velocity vector of test particle and $p^i = \frac{m_0 v^i}{\sqrt{1-v^2}}$ with $v^2 = \gamma_{np} v^n v^p$. Also, we know that the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ has components as $C^i = \frac{e^{ijk}}{\sqrt{\gamma}} A_j B_k$ in which $\gamma = \det(\gamma_{ij})$ and 3-dim Levi-Civita tensor e_{ijk} is antisymmetric in any exchange of indices while $e^{123} = e_{123} = 1$, [4, 6]. In (2), the time dependent gravitoelectromagnetism fields are defined in terms of the gravitoelectric and gravitomagnetic potentials through the following relations

$$ {}^*\mathbf{E}_g = -{}^*\nabla \Phi_g - \frac{\partial \mathbf{g}}{\partial t} \doteq {}^*E_{gi} = -{}^*\partial_i \Phi_g - \frac{\partial g_i}{\partial t}, \tag{3}$$

$$ {}^*\mathbf{B}_g = \sqrt{h}\, {}^*\nabla \times \mathbf{g} \doteq {}^*B^i_g = \frac{e^{ijk}}{2\sqrt{\gamma}}\, {}^*B_{jk}, \tag{4}$$

where $\Phi_g = \ln\sqrt{h}$, $\mathbf{g} = (g_1, g_2, g_3)$, ${}^*B_{ij} = \sqrt{h}\,(g_{j*i} - g_{i*j})$ such that $_{*i} = {}^*\partial_i = \partial_i + g_i\partial_0$ and curl of an arbitrary vector in a 3-space with time dependent metric γ_{ij} is defined by $({}^*\nabla \times \mathbf{A})^i = \frac{e^{ijk}}{2\sqrt{\gamma}}(A_{k*j} - A_{j*k})$. If we use the time dependent γ_{ij} as the metric tensor, then the Einstein field equations in vacuum case for this spacetime may be written as time dependent quasi-Maxwell equations,[3] [3, 7]:

$$ {}^*\nabla \cdot {}^*\mathbf{E}_g = {}^*E^2_g + \frac{1}{2}{}^*B^2_g - \frac{{}^*\partial D}{\partial t} - d, \tag{5}$$

$$ {}^*\nabla \times {}^*\mathbf{B}_g = 2({}^*\mathbf{E}_g \times {}^*\mathbf{B}_g + \mathbf{M}), \tag{6}$$

$$\begin{aligned} {}^*K_{ij} = &-{}^*\nabla_{(i}{}^*E_{gj)} + {}^*E_{gi}{}^*E_{gj} + \frac{1}{2}({}^*B_{gi}{}^*B_{gj} - \gamma_{ij}{}^*B^2_g) \\ &- DD_{ij} + 2D_{ik}D^k_j - {}^*B_{k(i}D^k_{j)} - \frac{{}^*\partial D_{ij}}{\partial t}, \end{aligned} \tag{7}$$

where the symbol () denotes the commutation over indices, ${}^*K_{ij}$ is 3-dim starry Ricci tensor constructed from 3-dim starry Christoffel symbols as ${}^*K_{ij} = {}^*\lambda^k_{ij*k} - {}^*\lambda^k_{ik*j} + {}^*\lambda^m_{ij}{}^*\lambda^k_{km} - {}^*\lambda^m_{ik}{}^*\lambda^k_{mj}$ in which ${}^*\lambda^i_{jk} = \frac{1}{2}\gamma^{ip}(\gamma_{jp*k} + \gamma_{kp*j} - \gamma_{jk*p})$ and starry covariant derivatives of an arbitrary 3-vector and a tensor are given respectively by the following familiar forms

$$ {}^*\nabla_j A_i = A_{i*j} - {}^*\lambda^k_{ij} A_k, \tag{8}$$

$$ {}^*\nabla_k T^{ij} = T^{ij}_{*k} + {}^*\lambda^i_{mk} T^{jm} + {}^*\lambda^j_{mk} T^{im}. \tag{9}$$

Also, $d = D_{ij}D^{ij}$ and $M^i = -{}^*\nabla_j D^{ij} + {}^*\partial^i D$ such that ${}^*\partial^i = \gamma^{in}\,{}^*\partial_n$ with

$$ D_{ij} = \frac{1}{2}\frac{{}^*\partial\gamma_{ij}}{\partial t}, \qquad D^{ij} = -\frac{1}{2}\frac{{}^*\partial\gamma^{ij}}{\partial t}, \qquad D = \frac{{}^*\partial \ln\sqrt{\gamma}}{\partial t}. \tag{10}$$

[3] In a 3-space with time dependent metric γ_{ij}, divergence of a vector is defined as ${}^*\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{\gamma}}(\sqrt{\gamma}A^i)_{*i}$.

2 The Exact Solution of Petrov Type {3, 1} Metric

The line element corresponding to the Petrov type {3, 1} is written as, [8, 9]:

$$ds^2 = dt^2 - n_2 t^{n_1} e^{-2z} dx^2 - t^{n_2} e^{4z} dy^2 - n_5 t^{n_3} dz^2 - 2n_6 t^{n_4} e^{-z} dx dz, \tag{11}$$

where $n_1, \ldots, n_6$ are unknown real constants. Firstly, a simple calculation shows that all components of gravitoelectromagnetism fields are zero. Also, we can deduce

$$D = \frac{n_2 + mp}{2t}, \tag{12}$$

$$(D_{ij}) = \frac{1}{2t} \begin{pmatrix} n_1 n_2 t^{n_1} e^{-2z} & 0 & n_4 n_6 t^{n_4} e^{-z} \\ 0 & n_2 t^{n_2} e^{4z} & 0 \\ n_4 n_6 t^{n_4} e^{-z} & 0 & n_3 n_5 t^{n_3} \end{pmatrix}, \tag{13}$$

$$(D^{ij}) = \frac{m}{2t} \begin{pmatrix} n_5 t^{n_3} e^{2z}(mp - n_3) & 0 & n_6 t^{n_4} e^{z}(n_4 - mp) \\ 0 & \frac{n_2}{m t^{n_2} e^{4z}} & 0 \\ n_6 t^{n_4} e^{z}(n_4 - mp) & 0 & n_2 t^{n_1}(mp - n_1) \end{pmatrix}, \tag{14}$$

where $m = \frac{1}{n_2 n_5 t^{n_1+n_3} - n_6^2 t^{2n_4}}$ and $p = (n_1 + n_3) n_2 n_5 t^{n_1+n_3} - 2n_4 n_6^2 t^{2n_4}$. In this step, with an elementary calculation, the nonvanishing components of 3-dim starry Christoffel symbols are determined as below

$$\begin{aligned}
{}^*\lambda^1_{11} &= -n_2 n_6 m t^{n_1+n_4} e^{-z}, \\
{}^*\lambda^1_{13} &= -n_2 n_5 m t^{n_1+n_3}, \\
{}^*\lambda^1_{22} &= 2n_6 m t^{n_2+n_4} e^{5z}, \\
{}^*\lambda^1_{33} &= -n_5 n_6 m t^{n_3+n_4} e^{z}, \\
{}^*\lambda^2_{23} &= 2, \\
{}^*\lambda^3_{11} &= n_2^2 m t^{2n_1} e^{-2z}, \\
{}^*\lambda^3_{13} &= n_2 n_6 m t^{n_1+n_4} e^{-z}, \\
{}^*\lambda^3_{22} &= -2n_2 m t^{n_1+n_2} e^{4z}, \\
{}^*\lambda^3_{33} &= n_6^2 m t^{2n_4}.
\end{aligned} \tag{15}$$

Then, with applying these symbols, the starry Ricci tensor can be calculated as follows

$${}^*K_{ij} = \begin{cases} n_2^2 m t^{2n_1} e^{-2z}, & i = j = 1, \\ -2n_2 m t^{n_1+n_2} e^{4z}, & i = j = 2, \\ n_2 n_6 m t^{n_1+n_4} e^{-z}, & i, j = 1, 3, \\ n_6^2 m t^{2n_4} - 5, & i = j = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

For later use, we will need the following components

$$M^1 = \frac{1}{2} n_6 m t^{n_4 - 1} e^{z}(mp - 2n_2), \tag{17}$$

$$M^2 = 0, \tag{18}$$

$$M^3 = n_2 m^2 t^{n_1-1}\left\{\frac{n_1+n_2}{m} - p + \frac{1}{2}n_2 n_5 t^{n_1+n_3}(n_3-n_1) + n_6^2 t^{2n_4}(n_1-n_4)\right\}. \quad (19)$$

On the other hand, the time dependent quasi-Maxwell equations reduce to

$$d + \frac{\partial D}{\partial t} = 0, \quad (20)$$

$$M^1 = 0, \quad (21)$$

$$M^3 = 0, \quad (22)$$

$$^*K_{ij} + DD_{ij} - 2\gamma^{nk} D_{ik} D_{nj} + \frac{\partial D_{ij}}{\partial t} = 0. \quad (23)$$

We now start with (21) and it means that $mp = 2n_2$ which implies that $n_2 = n_4$ and $n_1 + n_3 = 2n_2$. With applying previous results and using (16), we see that the (23), after a few simplifications, expand to

$$(i=j=1) \quad : \quad a_2 = (n_3-n_2)^2 a_1 + n_2\left(1 - \frac{3n_2}{2}\right), \quad (24)$$

$$(i=j=1,3) : \quad a_2 = (n_3-n_2)^2 a_1 - n_3(n_3+1) + 2n_2(1-2n_2) + \frac{7n_2 n_3}{2}, \quad (25)$$

$$(i=j=2) \quad : \quad a_2 = \frac{1}{4} n_2(3n_2-2), \quad (26)$$

$$(i=j=3) \quad : \quad a_1 = 6 + \frac{1}{2} n_5 t^{n_3-2}\left\{(n_3-n_2)^2(a_1-1) + n_3\left(1 - \frac{3n_2}{2}\right)\right\}. \quad (27)$$

In the above $a_1 = \frac{n_2 n_5}{n_2 n_5 - n_6^2}$ and $a_2 = \frac{2n_2 t^{2-n_3}}{n_2 n_5 - n_6^2}$. As a result, from (26) we can infer that $n_3 = 2$. Next, by comparing (24) and (25), one can derive $n_2 = \frac{6}{5}$ and $n_2 = 2$. But, it is easy to check that case $n_2 = 2$ yielding a contradiction. Therefore, we have $n_1 = \frac{2}{5}$ and $n_4 = \frac{6}{5}$. Then, from (26) we get that $a_2 = \frac{12}{25}$. Similarly, from (24) or (25) one learns that $a_1 = \frac{9}{4}$. Let us now replacing the value of a_1 into (27) and so leads to $n_5 = \frac{75}{8}$. Furthermore, with the help of definition a_1 or a_2, we conclude $n_6 = \frac{5}{2}$. Also, it can be shown that (20) and (22) are trivial. Finally, we can rewrite the metric (11) as follows

$$ds^2 = dt^2 - \frac{6}{5} t^{\frac{2}{5}} e^{-2z} dx^2 - t^{\frac{6}{5}} e^{4z} dy^2 - \frac{75}{8} t^2 dz^2 - 5t^{\frac{6}{5}} e^{-z} dx dz. \quad (28)$$

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